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Stratification of the Null Cone in the Non-split Case

by

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Dedicated to Professor Fumihiro Sato on the occasion of his 65 th birthday

Abstract. In early 80's, the notion of stratification of the null cone of reductive group actions was studied by Kempf, Ness and Kirwan. Their work includes stratifications of finite dimensional representations of reductive groups. Even though they did not explicitly stated, their construction was rational over a perfect ground field k as long as the group is split over k. In this paper, we extend these stratifications for all (not necessarily split) reductive groups over k.

1. Introduction

Let k be a field. For any algebraic group G over k, a homomorphism from $\mathbb{G}_m = \mathrm{GL}_1$ to G is called a one parameter subgroup (which will be abbreviated as 1-PS from now on). Let $X^*(G) = \mathrm{Hom}(G, \mathbb{G}_m)$ and $X_*(G) = \mathrm{Hom}(\mathbb{G}_m, G)$ be the group of characters of G and the group of 1-PS's of G (defined over the algebraic closure $\bar{k}$) respectively. Also let $X^*_k(G) = \mathrm{Hom}_k(G, \mathbb{G}_m)$ and $X_{*,k}(G) = \mathrm{Hom}_k(\mathbb{G}_m, G)$ be the group of rational characters of G and the group of rational 1-PS's of G respectively.

Let G be a connected reductive algebraic group and V a representation of G both defined over k. Choose a maximal k-split torus S of G and a maximal torus T of G defined over k containing S ([1, 18.2]). We put

$$\mathfrak{s} = X_*(S) \otimes \mathbb{R} = X_{*,k}(S) \otimes \mathbb{R}\,, \quad \mathfrak{s}^* = X^*(S) \otimes \mathbb{R} = X^*_k(S) \otimes \mathbb{R}\,,$$
$$\mathfrak{t} = X_*(T) \otimes \mathbb{R}\,, \quad \mathfrak{t}^* = X^*(T) \otimes \mathbb{R}\,.$$

Since T is defined over k, the Galois group $\mathrm{Aut}_k\,\bar{k}$ acts on $\mathfrak{t}, \mathfrak{t}^*$. We also put $\mathfrak{s}_{\mathbb{Q}} = X_*(S) \otimes \mathbb{Q}$, etc. Let $\mathbb{W} = N_G(T)/T$, ${}_k\mathbb{W} = N_G(S)/Z_G(S)$ be the Weyl group of G and the relative Weyl group of G respectively.

There is a natural pairing $\langle\ ,\ \rangle_T : X^*(T) \times X_*(T) \to \mathbb{Z}$ defined by $t^{\langle \chi,\lambda\rangle} = \chi(\lambda(t))$ for $\chi \in X^*(T), \lambda \in X_*(T)$. This is a perfect paring ([1, pp.113–115]). Similarly, there is a perfect pairing $\langle\ ,\ \rangle_S : X^*(S) \times X_*(S) \to \mathbb{Z}$.

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There exists an inner product $(\ ,\ )_{\mathfrak{t}}$ on $\mathfrak{t}$ which is invariant under the action of $\mathbb{W}$. Since $\mathrm{Aut}_k\,\bar{k}$ leaves $N_G(T)$ invariant, we may assume that this inner product is invariant under the action of $\mathrm{Aut}_k\,\bar{k}$. We may assume that this inner product is rational, i.e., $(\lambda, \nu)_{\mathfrak{t}} \in \mathbb{Q}$ for all $\lambda, \nu \in \mathfrak{t}_{\mathbb{Q}}$. We denote the inner product on $\mathfrak{s}$ obtained by restricting $(\ ,\ )_{\mathfrak{t}}$ to $\mathfrak{s}$ by $(\ ,\ )_{\mathfrak{s}}$. It is easy to see that $(\ ,\ )_{\mathfrak{s}}$ is also rational. Any element of ${}_k\mathbb{W}$ is represented by an element of $N_G(T)$ (Lemma 4.1). In fact ${}_k\mathbb{W}$ can be regarded as a subgroup of $\mathbb{W}$ ([2, 5.5.Corollaire.]). Therefore, $(\ ,\ )_{\mathfrak{s}}$ is invariant by the action of ${}_k\mathbb{W}$. Let $\|\ \|_{\mathfrak{s}}$ (resp. $\|\ \|_{\mathfrak{t}}$) be the norm on $\mathfrak{s}$ (resp. $\mathfrak{t}$) defined by $(\ ,\ )_{\mathfrak{s}}$ (resp. $(\ ,\ )_{\mathfrak{t}}$). We choose a Weyl chamber $\mathfrak{s}_+ \subset \mathfrak{s}$ (resp. $\mathfrak{t}_+ \subset \mathfrak{t}$) for the action of ${}_k\mathbb{W}$ (resp. $\mathbb{W}$).

For $\lambda \in \mathfrak{s}$, let $\beta = \beta(\lambda)$ be the element of $\mathfrak{s}^*$ such that $\langle \beta, \nu \rangle_S = (\lambda, \nu)_{\mathfrak{s}}$ for all $\nu \in \mathfrak{s}$. The map $\lambda \mapsto \beta(\lambda)$ is a bijection and we denote the inverse map by $\lambda = \lambda(\beta)$. We have a similar bijection between $\mathfrak{t}$ and $\mathfrak{t}^*$. We use the same notation $\beta(\lambda)$, $\lambda(\beta)$ for this bijection. There is a unique positive rational number a such that $a\lambda(\beta) \in X_*(S)$ or $X_*(T)$ and is indivisible. We use the notation λ_β for $a\lambda(\beta)$.

Identifying $\mathfrak{s}$ (resp. , $\mathfrak{t}$) with $\mathfrak{s}^*$ (resp. $\mathfrak{t}^*$), we have a ${}_k\mathbb{W}$-invariant (resp. $\mathbb{W}$-invariant) inner product $(\ ,\ )_{\mathfrak{s}^*}$ (resp. $(\ ,\ )_{\mathfrak{t}^*}$) on $\mathfrak{s}^*$ (resp. $\mathfrak{t}^*$), the norm $\|\ \|_{\mathfrak{s}^*}$ (resp. $\|\ \|_{\mathfrak{t}^*}$) determined by $(\ ,\ )_{\mathfrak{s}^*}$ (resp. $(\ ,\ )_{\mathfrak{t}^*}$) and a Weyl chamber $\mathfrak{s}^*_+$ (resp. $\mathfrak{t}^*_+$).

We recall the definition of stability over $\bar{k}$. Let $\pi_V : V \setminus \{0\} \to \mathbb{P}(V)$ be the natural projection map. Let $\bar{k}[V]^{G_{\bar{k}}}$ be the ring of polynomials invariant under the action of $G_{\bar{k}}$. Suppose that $P \in \bar{k}[V]^{G_{\bar{k}}} \setminus \bar{k}$ is a homogeneous polynomial. We define $\mathbb{P}(V)_P = \{\pi(v) \mid P(v) \neq 0\}$.

DEFINITION 1.1. Let $x \in \mathbb{P}(V)_{\bar{k}}$.

(1) x is semistable if there exists a homogeneous polynomial $P \in \bar{k}[V]^{G_{\bar{k}}} \setminus \bar{k}$ such that $x \in \mathbb{P}(V)_P$,

(2) x is unstable if it is not semistable.

We use the notation $\mathbb{P}(V)^{\mathrm{ss}}_{\bar{k}}$ for the set of semistable points. This is an $\mathrm{Aut}_k\,\bar{k}$-invariant open subset in $\mathbb{P}(V)_{\bar{k}}$. Also if $x = \pi_V(v) \in \mathbb{P}(V)_k \cap \mathbb{P}(V)^{\mathrm{ss}}_{\bar{k}}$ then there exists $P \in \bar{k}[V]^{G_{\bar{k}}} \cap (k[V] \setminus k)$ such that $P(v) \neq 0$. Therefore, the notion of semistability is rational over the ground field.

Let λ be a non-trivial 1-PS of G over $\bar{k}$. Suppose that $v \in V \setminus \{0\}$, $\pi_V(v) = x$ and $v = \sum_{i=1}^n v_i$ is the eigen decomposition with respect to λ, i.e., $\lambda(t)v = \sum_{i=1}^n t^{r_i} v_i$, $v_i \neq 0$ for all i, and $r_i \neq r_j$ if $i \neq j$. Then we define a numerical function by $\mu(x, \lambda) = \min_{1 \le i \le n} r_i$. For later purposes, we would like to define $\mu(x, \lambda)$ for $\lambda \in \mathfrak{t}_{\mathbb{Q}} \setminus \{0\}$. For that, if $m > 0$ is a positive integer and $\nu = m\lambda$ (written additively) is an element of $X_*(T)$, then we define $\mu(x, \lambda) = (1/m)\mu(x, \nu)$. This definition is apparently well-defined.

THEOREM 1.2 (Hilbert–Mumford criterion of stability [10]). *Let $x \in \mathbb{P}(V)_{\bar{k}}$, then x is semistable if and only if $\mu(x, \lambda) \le 0$ for all non-trivial 1-PS's λ.*

Note that the above statement is equivalent to the statement that x is unstable if and only if $\mu(x, \lambda) > 0$ for a non-trivial 1-PS λ. Also there is a definition for a point to be "properly stable" but we shall not need it in this paper.

Since S is a split torus, its action is diagonalizable over the ground field k. So we choose a coordinate system $v = (v_0, v_1, \ldots, v_N)$ on V by which S acts diagonally. Let

$\gamma_i \in \mathfrak{s}^*$ and e_i be the weight and the coordinate vector which corresponds to i-th coordinate. For a subset $\mathfrak{I} \subset \{\gamma_i \mid i = 0, 1, \dots, N\}$, we denote the convex hull of $\mathfrak{I}$ by $\operatorname{Conv}\mathfrak{I}$. If $v \in V \setminus \{0\}$ and $x = \pi_V(v)$ then we put $\mathfrak{I}_v = \mathfrak{I}_x = \{\gamma_i \mid v_i \neq 0\}$.

For $\mathfrak{I} \subset \{\gamma_i \mid i = 0, 1, \dots, N\}$ such that $0 \notin \operatorname{Conv}\mathfrak{I}$, let β be the closest point of $\operatorname{Conv}\mathfrak{I}$ to the origin. Then β lies in $\mathfrak{s}^*_{\mathbb{Q}}$. Note that $(\xi, \beta)_{\mathfrak{s}^*} \geq (\beta, \beta)_{\mathfrak{s}^*}$ holds for all $\xi \in \operatorname{Conv}\mathfrak{I}$ since $\operatorname{Conv}\mathfrak{I}$ is convex. Let $\mathfrak{B}$ be the set of all such β which lie in $\mathfrak{s}^*_+$.

We define

$$
\begin{aligned}
&Y_\beta = \operatorname{span}\{e_i \mid (\gamma_i, \beta)_{\mathfrak{s}^*} \geq (\beta, \beta)_{\mathfrak{s}^*}\}, \quad Z_\beta = \operatorname{span}\{e_i \mid (\gamma_i, \beta)_{\mathfrak{s}^*} = (\beta, \beta)_{\mathfrak{s}^*}\}, \\
&W_\beta = \operatorname{span}\{e_i \mid (\gamma_i, \beta)_{\mathfrak{s}^*} > (\beta, \beta)_{\mathfrak{s}^*}\}.
\end{aligned}
$$

Clearly $Y_\beta = Z_\beta \oplus W_\beta$.

If λ is a 1-PS of G, we define

$$
\begin{aligned}
P(\lambda) &= \left\{ p \in G \;\middle|\; \lim_{t \to 0} \lambda(t) p \lambda(t)^{-1} \text{ exists} \right\}, \quad M(\lambda) = Z_G(\lambda), \\
U(\lambda) &= \left\{ p \in G \;\middle|\; \lim_{t \to 0} \lambda(t) p \lambda(t)^{-1} = 1 \right\}.
\end{aligned}
$$

The group $P(\lambda)$ is a parabolic subgroup of G ([14, p.148]) with Levi part $M(\lambda)$ and unipotent radical $U(\lambda)$. We put $P_\beta = P(\lambda_\beta)$, $M_\beta = Z_G(\lambda_\beta)$ and $U_\beta = U(\lambda_\beta)$.

Let χ_β be the indivisible rational character of M_β such that the restriction of χ_β^a to S coincides with $b\beta$ for some positive integers a, b. We define $G_\beta = \{g \in M_\beta \mid \chi_\beta(g) = 1\}^\circ$ (the identity component). Then G_β acts on Z_β. Note that M_β and G_β are defined over k, and since $\langle \chi_\beta, \lambda_\beta \rangle$ is a positive multiple of $\|\beta\|_{\mathfrak{s}^*}$, $M_\beta = G_\beta \lambda_\beta$. Moreover, if ν is any rational 1-PS in G_β, $(\nu, \lambda_\beta)_{\mathfrak{s}} = 0$.

Let $\mathbb{P}(Z_\beta)^{\mathrm{ss}}$ be the set of G_β-semistable points of $\mathbb{P}(Z_\beta)$. We regard $\mathbb{P}(Z_\beta)^{\mathrm{ss}}$ as a subset of $\mathbb{P}(V)$. Put

$$
\begin{aligned}
&Z_\beta^{\mathrm{ss}} = \pi_V^{-1}(\mathbb{P}(Z_\beta)^{\mathrm{ss}}), \quad Y_\beta^{\mathrm{ss}} = \{(z, w) \mid z \in Z_\beta^{\mathrm{ss}}, w \in W_\beta\}, \\
&\mathbb{P}(Y_\beta)^{\mathrm{ss}} = \{\pi_V((z, w)) \mid (z, w) \in Y_\beta^{\mathrm{ss}}\}.
\end{aligned}
$$

We define $S_\beta = G Y_\beta^{\mathrm{ss}}$. Note that S_β can be the empty set. We denote the set of k-rational points of S_β, etc., by $S_{\beta\, k}$, etc.

The following theorem is the main result of this paper.

THEOREM 1.3. *Suppose that k is a perfect field. Then we have*

$$
V_k \setminus \{0\} = V_k^{\mathrm{ss}} \amalg \coprod_{\beta \in \mathfrak{B}} S_{\beta\, k}.
$$

Moreover, $S_{\beta\, k} \cong G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}}$.

We remind the reader that $G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}}$ means $(G_k \times Y_{\beta\, k}^{\mathrm{ss}})/\sim$ where $(g, v), (g'v') \in G_k \times Y_{\beta\, k}^{\mathrm{ss}}$ are equivalent if there exists an element $p \in P_{\beta\, k}$ such that $g' = gp^{-1}$ and $v' = pv$. In Theorem 1.3, the bijection $S_{\beta\, k} \cong G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}}$ is induced from the canonical map $G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}} \ni (g, v) \mapsto gv \in S_{\beta\, k}$.

If G is a split reductive group, Theorem 1.3 was proved in principle by Kirwan [8], Ness [11]. The main purpose of [8] was calculating equivariant cohomology groups by using the equivariant morse stratification. Kirwan used the inductive structure of the strata for that purpose. On the other hand, in [11], Ness studied the stratification of the null cone depending more on geometric invariant theory. She studied the stratification of the null cone by the convexity of the moment map. The rationality of the stratification is basically due to the earlier work of Kempf [7].

We intend to study zeta functions associated to prehomogeneous vector spaces using the stratification as in [17]. For that purpose, we need a completely algebraic approach and the rationality of the inductive structure. Also in number theoretic situations, there are interesting non-split groups such as orthogonal groups, unitary groups, restrictions of scalars, etc. So even though our proof is fairly easy if we assume the split case, it is probably worth pointing out how the non-split case is reduced to the split case. This is the main purpose of this paper.

We fix a perfect field k. The representation of $G = \mathrm{GL}_2$ on $V = \mathrm{Sym}^3\mathrm{Aff}^2$ (over k) is an example of what we call a "prehomogeneous vector space", where Aff^2 denotes the two dimensional affine space which is regard as a vector space of dimension two (similarly, we use the notation Aff^n for the n dimensional affine space). For the notion of prehomogeneous vector spaces, see [9] or [13]. In this situation we are interested in G_k-orbits in V_k. However, if we are to use Theorem 1.3 in this situation, we have to consider the action of SL_2 on V instead of GL_2. So we would like to modify Theorem 1.3 so that it is applicable to the action of the groups which correspond to GL_2 in this situations.

Let G be a reductive group and V a representation of G both defined over k. We assume that there is a reductive subgroup G_1 of G, a torus $T_0 \subset Z(G)$ (the center of G) with positive split rank and a rational character χ of T_0 such that $T_0 \cap G_1$ is finite and that $G = T_0 G_1$ as algebraic groups (i.e., $G_{\bar{k}} = T_{0\bar{k}} G_{1\bar{k}}$). We also assume that the action of $t \in T_0$ on V is given by the scalar multiplication by $\chi(t)$. Let S be a maximal split torus of G_1 (this is the difference from the situation of Theorem 1.3) and we define $\mathfrak{s}^*$, $\mathfrak{s}^*_+$, $\mathfrak{B}$, etc., with respect to the group G_1. For $\beta \in \mathfrak{B}$, we define Z_β, W_β, Y_β, Y^{ss}_β, etc., as in Theorem 1.3 with respect to G_1 also. Then we have the following corollary.

COROLLARY 1.4. *In the above situation, the statement of Theorem* 1.3 *holds.*

This paper is organized as follows. In section 2, we recall Kempf's result which will be used in sections 3, 4. The rationally property of Kempf's result will be used mainly in section 4. In section 3, we give a proof of Theorem 1.3 in the case where the ground field is algebraically closed (and so the group is split). This case is in principle known. However, we are interested in the rationality question and we use a formalization which is a combination of methods in [11], [8] so that it is easier to deduce the rationality result. Therefore, we included the outline of the proof of this case. We give a proof of Theorem 1.3 and Corollary 1.4 in the non-split case in section 4. Finally, in section 5, we give two series of prehomogeneous vector spaces which have the same sets of weights with respect to the action of maximal split tori. One is similar to the space of pairs of ternary quadratic forms and the other is slightly easier and similar to the space of pairs of binary quadratic forms.

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2. Kempf's result

First we assume that $k = \bar{k}$ and so $S = T$. Let X be a variety over k and $f : \mathbb{G}_m \to X$ a morphism . We embed $\mathbb{G}_m$ to the one dimensional affine space Aff^1 in the usual manner. We say that $\lim_{t \to 0} f(t) = y$ if there exists a morphism $g : \mathrm{Aff}^1 \to X$ such that $g|_{\mathbb{G}_m} = f$ and $g(0) = y$.

Let $v \in V \setminus \{0\}$ and $x = \pi_V(v) \in \mathbb{P}(V)$. We define

$$|V, v| = \left\{ \lambda \in X_*(G) \;\middle|\; \lim_{t \to 0} \lambda(t)v \text{ exists in } V \right\},$$

$$|V, v|_{\{0\}} = \left\{ \lambda \in X_*(G) \;\middle|\; \lim_{t \to 0} \lambda(t)v = 0 \right\}.$$

If λ is any 1-PS of G then it is conjugate to an element γ of $X_*(T)$. If γ_1 is another such element of $X_*(T)$ then γ, γ_1 are conjugate by an element of $\mathbb{W}$. Since the norm is invariant by the action of $\mathbb{W}$, $\|\gamma\|_{\mathfrak{t}}$ depends only on λ. So we define $\|\lambda\| = \|\gamma\|_{\mathfrak{t}}$.

The following theorem is Theorem 3.4 in Kempf [7].

THEOREM 2.1 (Kempf [7]). (1) *The function $\mu(x, \lambda)/|\|\lambda\|$ has a maximal value* (*which we denote by $M(x)$*) *on the set $|V, v|$ if it is not empty.*

(2) *The condition $\overline{Gv} \ni 0$ is equivalent to the condition that $|V, v|_{\{0\}}$ is not empty.*

(3) *Suppose that $|V, v|_{\{0\}}$ is not empty, and let Λ_x be the set of indivisible 1-PS λ's such that $\mu(x, \lambda) = M(x)\|\lambda\|$.*
   - (a) *The set Λ_x is non empty, and there exists a parabolic subgroup P_x of G such that $P_x = P(\lambda)$ for all $\lambda \in \Lambda_x$.*
   - (b) *The set Λ_x is a principal homogeneous space under the action of the unipotent radical of P_x.*
   - (c) *Any maximal torus of P_x contains a unique element of Λ_x.*

REMARK 2.2. The condition $\overline{Gv} \ni 0$ holds if and only if x is unstable.

Next we recall the rationality property of Kempf's result. This property of Kempf's result will be used mainly in section 4. The next theorem is Theorem 4.2 in Kempf [7].

THEOREM 2.3 (Kempf [7]). *Suppose that $v \in V_k$, and that $|V, v|_{\{0\}\,\bar{k}}$ is not empty. Put $x = \pi_V(v)$. Then all elements in Λ_x are split, P_x is rationally conjugate to a standard parabolic subgroup, and Λ_x is a principal homogeneous space under the action of the k-points of the unipotent radical of P_x.*

This theorem implies that if $v \in V_k$ and $|V, v|_{\{0\}\,\bar{k}}$ is not empty, v is conjugate to $v' \in V_k$ by an element of G_k and Λ_y contains a split 1-PS of T, where $y = \pi_V(v')$.

Finally, we introduce the terminology of "adapted 1-PS" which we will be use in section 3, 4.

DEFINITION 2.4. We say that a 1-PS λ is G-adapted (or adapted, simply) for x if $M(x)\|\lambda\| = \mu(x, \lambda)$.

3. Proof of the main theorem in the split case

We first assume that $k = \bar{k}$ (so $S = T$) and so we shall not use the subscript $_k$ until we consider the rationality questions.

The next lemma is proved in [10, p. 57, Proposition 2.7].

LEMMA 3.1. *Let ξ be a 1-PS of G. Then $\mu(qx, \xi) = \mu(x, \xi)$ holds for all $q \in P(\xi)$.*

The next lemma is proved in [11, Lemma 9.2]. However, since there are minor inaccuracies in the proof, we give a full proof here.

LEMMA 3.2. *Let $x = \pi_V((z, w)) \in \mathbb{P}(Y_\beta)$ where $z \in Z_\beta \setminus \{0\}$ and $w \in W_\beta$. We put $M = \mu(x, \lambda_\beta)/\|\lambda_\beta\| > 0$. Then $\pi_V(z)$ is G_β-unstable if and only if $\mu(x, \xi)/\|\xi\| > M$ for some $\xi \in X_*(P_\beta)$.*

Proof. We first show that if $\pi_V(z)$ is G_β-semistable then $\mu(x, \xi)/\|\xi\| \le M$ for all $\xi \in X_*(P_\beta)$.

Since ξ is conjugate to a 1-PS of M_β, there exists $p \in P_\beta$ such that $\xi_1 \overset{\text{def}}{=} p\xi p^{-1} \in X_*(M_\beta)$. If we write $p = mu_1$ where $m \in M_\beta$ and $u_1 \in U_\beta$ then

$$\xi = p^{-1}\xi_1 p = u_1^{-1}m^{-1}\xi_1 m u_1 = m^{-1}\xi_1 m u$$

where $u = (m^{-1}\xi_1 m)^{-1}u_1^{-1}(m^{-1}\xi_1 m)u_1 \in U_\beta$. If we put $\eta = m^{-1}\xi_1 m$, then $\eta \in X_*(M_\beta)$ is conjugate to ξ by an element of P_β. Therefore, $\|\xi\| = \|\eta\|$.

Since $u(t)((z, w)) = (z, w')$ where $w' = w'(t) \in W_\beta$, $\xi((z, w)) = (\eta z, \eta w')$. There exist $m, n \in \mathbb{Z}$ and $\nu \in X_*(G_\beta)$ such that $m > 0$ and $\eta^m = \nu\lambda_\beta^n$. Now $\lambda_\beta(t)$ acts on Z_β by scalar multiplication, say $\lambda_\beta(t)z = t^a z$ where $a > 0$. We choose a coordinate system $z = (z_i)$ of Z_β so that $\nu(t)z = (t^{b_i}z_i)$. Since $\pi_V(z)$ is G_β-semistable, there is i such that $z_i \neq 0$ and $b_i \le 0$. Then the i-th component of $\eta^m(t)z$ is $t^{na+b_i}z_i$. Since λ_β and ν are orthogonal,

$$\|\eta^m\| = \sqrt{\|\nu\|^2 + |n|^2\|\lambda_\beta\|^2} > |n|\|\lambda_\beta\|\,.$$

So,

$$\frac{\mu(x, \xi)}{\|\xi\|} = \frac{\mu(x, \eta^m)}{\|\eta^m\|} \le \frac{na + b_i}{\|\eta^m\|} \le \frac{na}{|n|\|\lambda_\beta\|} \le \frac{a}{\|\lambda_\beta\|} = M\,.$$

Conversely we assume that $\mu(x, \nu) > 0$. As above we assume $\nu(t)z = (t^{b_i}z_i)$. We choose a coordinate system $w = (w_j)$ so that $\lambda_\beta(t)w = (t^{c_j}w_j)$ and $\nu(t)w = (t^{d_j}w_j)$. Then $c_j > a$ and

$$(\nu\lambda_\beta^n)(t)w = (t^{nc_j + d_j}w_j)\,, \quad (\nu\lambda_\beta^n)(t)z = (t^{na+b_i}z_i)\,.$$

Since there are finitely many possibilities for i, j, if n is sufficiently large, $nc_j + d_j > na + b_i$ for all i, j. Therefore, $\mu(x, \nu\lambda_\beta^n) = \mu(x, \nu) + na = \mu(x, \nu) + n\mu(x, \lambda_\beta)$. So,

$$\frac{\mu(x, \nu\lambda_\beta^n)}{\|\nu\lambda_\beta^n\|} = \frac{\mu(x, \nu) + n\mu(x, \lambda_\beta)}{\sqrt{\|\nu\|^2 + |n|^2\|\lambda_\beta\|^2}}.$$

Put $b = \mu(x, \nu) > 0$. Then, we would like to prove

$$\frac{b + na}{\sqrt{\|\nu\|^2 + |n|^2\|\lambda_\beta\|^2}} > \frac{a}{\|\lambda_\beta\|} = M \tag{3.3}$$

for sufficiently large $n > 0$. Clearly the inequality

$$\begin{aligned} &\|\lambda_\beta\|^2(b^2 + 2abn + n^2a^2) - a^2(\|\nu\| + n^2\|\lambda_\beta\|^2) \\ &= \|\lambda_\beta\|^2(b^2 + 2abn) - a^2\|\nu\|^2 > 0 \end{aligned}$$

is valid for sufficiently large $n > 0$, and so

$$\frac{b^2 + 2nab + n^2a^2}{\|\nu\|^2 + n^2\|\lambda_\beta\|^2} > \frac{a^2}{\|\lambda_\beta\|^2}\,. \tag{3.4}$$

Therefore, (3.3) follows. □

PROPOSITION 3.5. *Assume that $x \in \mathbb{P}(Y_\beta)$. Then $x \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ if and only if λ_β is adapted for x and $\mu(x, \lambda_\beta)/\|\lambda_\beta\| = \|\beta\|_{\mathfrak{t}^*}$.*

Proof. We show the "if" part first. Suppose that $x = \pi_V(v)$ and $v = (v_0, \ldots, v_N) = (z, w)$ where $z \in Z_\beta$ and $w \in W_\beta$. Since $\lambda(\beta) = a\lambda_\beta$ for a positive rational number a,

$$\frac{\mu(x, \lambda_\beta)}{\|\lambda_\beta\|} = \frac{\min_{v_i \neq 0}\langle \gamma_i, \lambda_\beta\rangle_T}{\|\lambda_\beta\|} = \frac{\min_{v_i \neq 0}\langle \gamma_i, \lambda(\beta)\rangle_T}{\|\lambda(\beta)\|} = \frac{\min_{v_i \neq 0}(\gamma_i, \beta)_{\mathfrak{t}^*}}{\|\beta\|_{\mathfrak{t}^*}}\,. \tag{3.6}$$

By assumption, $\min_{v_i \neq 0}(\gamma_i, \beta)_{\mathfrak{t}^*} = \|\beta\|^2_{\mathfrak{t}^*}$. So $z \neq 0$. Since λ_β is adapted for x, $x \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ by Lemma 3.2.

We next show the "only if" part. Suppose that $x = \pi((z, w)) \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ where $z \in Z_\beta$, $w \in W_\beta$. Since $z \neq 0$, by (3.6), $M \stackrel{\mathrm{def}}{=} \mu(x, \lambda_\beta)/\|\lambda_\beta\| = \|\beta\|_{\mathfrak{t}^*}$. By Lemma 3.2, $\mu(x, \xi)/\|\xi\| \leq M$ holds for all $\xi \in X_*(P_\beta)$.

We want to prove that λ_β is adapted for x. So we must show that

$$M = \frac{\mu(x, \lambda_\beta)}{\|\lambda_\beta\|} = \sup_{\xi \in X_*(G)} \frac{\mu(x, \xi)}{\|\xi\|}\,.$$

Since x is unstable, there exists an adapted 1-PS ξ for x by Theorem 2.1 (1). Since the intersection of any two parabolic subgroups contains a maximal torus, there is a maximal torus $T' \subset P_\beta \cap P(\xi)$. Since any torus in either P_β or $P(\xi)$ is conjugate to a subtorus of T', there exist $p \in P_\beta$ and $q \in P(\xi)$ such that $p\lambda_\beta p^{-1} = \lambda'_\beta$ and $q\xi q^{-1} = \xi'$ are both in $X_*(T')$.

Since $q^{-1} \in P(\xi) = P(\xi')$, by Lemma 3.1,

$$\mu(x, \xi) = \mu(qx, q\xi q^{-1}) = \mu(q^{-1}qx, q\xi q^{-1}) = \mu(x, \xi')\,.$$

Therefore, ξ' is also adapted for x, and so $M(x) = \mu(x, \xi')/\|\xi'\|$. Also ξ' is a 1-PS in $P(\xi')$, and so $M(x) = \mu(x, \xi')/\|\xi'\| \leq M$ by Lemma 3.2. Clearly $M(x) \geq M$ holds, and so $M(x) = M$. Therefore, λ_β is adapted for x. □

Proof of Theorem 1.3 (*split case*). Let $v \in V \setminus \{0\}$ and $x = \pi_V(v)$ be unstable. We would like to show that there is $\beta \in \mathfrak{B}$ such that $x \in G\mathbb{P}(Y_\beta)^{\mathrm{ss}}$. There is a 1-PS λ for which x is adapted. We choose $g \in G$ so that $\nu = g\lambda(t)g^{-1}$ is a 1-PS in T and that $\nu \in \mathfrak{t}_+$.

Then gx is ν-adapted. So we may assume that λ is a 1-PS in T such that $\lambda \in \mathfrak{t}_+$ to begin with.

Let $a = M(x)/\|\lambda\|$. Then

$$\frac{\mu(x, a\lambda)}{a^2\|\lambda\|^2} = \frac{\mu(x,\lambda)}{M(x)\|\lambda\|} = 1\,.$$

So by replacing λ by $a\lambda$, we may assume that $\mu(x,\lambda) = \|\lambda\|^2$. This λ may no longer be a 1-PS, but is an element of $\mathfrak{t}_{\mathbb{Q}}$. Let $\beta = \beta(\lambda) \in \mathfrak{t}^*_{\mathbb{Q}}$. Then $M(x) = \|\lambda\| = \|\beta\|_{\mathfrak{t}^*}$ and $\mu(x,\lambda) = \|\beta\|^2_{\mathfrak{t}^*}$.

As before, let $v = (v_i)$ be the coordinate of v and γ_i the weight of the i-th coordinate. Since $\mu(x,\lambda) = \|\beta\|^2_{\mathfrak{t}^*}$,

$$\langle \gamma_i, \lambda\rangle_T = (\gamma_i, \beta)_{\mathfrak{t}^*} \geq \|\beta\|^2_{\mathfrak{t}^*}$$

for all i and there exists i such that $(\gamma_i,\beta)_{\mathfrak{t}^*} = \|\beta\|^2_{\mathfrak{t}^*}$. So $v \in Y_\beta$ and if we write $v = (z, w)$ where $z \in Z_\beta$ and $w \in W_\beta$, then $z \neq 0$. Since $M(x) = \mu(x,\lambda)/\|\lambda\|$, $\pi_V(z)$ is G_β-semistable by Lemma 3.2. This implies that $x \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$. Also $\mu(x,\lambda_\beta)/\|\lambda_\beta\| = \mu(x,\lambda)/\|\lambda\| = \|\beta\|_{\mathfrak{t}^*}$.

We show that this β belongs to $\mathfrak{B}$. Put $T_\beta = T \cap G_\beta$. Define $\mathfrak{t}^*_\beta = X^*(T_\beta) \otimes \mathbb{R}$. Let $\beta^\perp = \{\gamma \in \mathfrak{t}^* \mid (\gamma,\beta)_{\mathfrak{t}^*} = 0\}$. Then $\beta^\perp \cong \mathfrak{t}^*_\beta$ by the natural homomorphism $\mathfrak{t}^* \to \mathfrak{t}^*_\beta$ and $\mathfrak{t}^* = \beta^\perp \oplus \mathbb{R}\beta$.

Let $z = (z_j)$ be a coordinate system of Z_β such that the action of T is diagonalized. Let δ_j be the weight of the j-th coordinate of z. Since $(\delta_j,\beta)_{\mathfrak{t}^*} = (\beta,\beta)_{\mathfrak{t}^*}$, $\delta'_j \overset{\mathrm{def}}{=} \delta_j - \beta \in \beta^\perp$ for all j. Also, let $w = (w_l)$ be a coordinate system of W_β such that the action of T is diagonalized. Let ε_l be the weight of the l-th coordinate of w.

Let $v = (z, w)$, $I_z = \{j \mid z_j \neq 0\}$ and $I_w = \{l \mid w_l \neq 0\}$. Since $\pi_V(z)$ is G_β-semistable and $\beta^\perp \cong \mathfrak{t}^*_\beta$, the convex hull of $\{\delta'_j \mid j \in I_z\}$ contains the origin. This means that there is $a_j \in \mathbb{R}$ for all $j \in I_z$ such that $0 \leq a_j \leq 1$, $\sum_{j\in I_z} a_j = 1$ and $\sum_{j\in I_z} a_j\delta'_j = 0$. Therefore,

$$\sum_{j\in I_z} a_j\delta_j = \beta\,.$$

So β belongs to the convex hull of $\{\delta_j \mid j \in I_z\}$. Obviously, β belongs to the convex hull of $\{\delta_j \mid j \in I_z\} \cup \{\varepsilon_l \mid l \in I_w\}$.

Suppose that $b_j \in \mathbb{R}$ $(j \in I_z)$, $c_l \in \mathbb{R}$ $(l \in I_w)$, $0 \leq b_j \leq 1$, $0 \leq c_l \leq 1$, $\sum_{j\in I_z} b_j + \sum_{l\in I_w} c_j = 1$ and

$$\alpha = \sum_j b_j\delta_j + \sum_l c_l\varepsilon_l.$$

Since $(\varepsilon_l,\beta)_{\mathfrak{t}^*} > (\beta,\beta)_{\mathfrak{t}^*}$, $(\varepsilon_l,\beta)_{\mathfrak{t}^*} = d_l(\beta,\beta)_{\mathfrak{t}^*}$ where $d_l > 1$. We put $\varepsilon'_l = \varepsilon_l - d_l\beta$. Then $\varepsilon'_l \in \beta^\perp$. We put

$$\alpha' = \sum_j b_j\delta'_j + \sum_l c_l\varepsilon'_l.$$

Then $\alpha' \in \beta^{\perp}$ and

$$\alpha = \left(\sum_j b_j + \sum_l c_l d_l \right) \beta + \alpha' .$$

We put $C = \sum_j b_j + \sum_l c_l d_l \geq \sum_j b_j + \sum_l c_l = 1$. Then

$$(\alpha, \alpha)_{\mathfrak{t}^*} = C^2 (\beta, \beta)_{\mathfrak{t}^*} + (\alpha', \alpha')_{\mathfrak{t}^*} \geq (\beta, \beta)_{\mathfrak{t}^*} + (\alpha', \alpha')_{\mathfrak{t}^*} \geq (\beta, \beta)_{\mathfrak{t}^*} .$$

Therefore, β is the closest point to the origin of the convex hull of weights of non-zero coordinates of $v = (z, w)$. Since $\beta \in \mathfrak{t}^*_+$, $\beta \in \mathfrak{B}$. This proves that

$$V \setminus \{0\} = V^{\mathrm{ss}} \cup \bigcup_{\beta \in \mathfrak{B}} S_\beta . \tag{3.7}$$

Suppose that $g_1, g_2 \in G$, $\beta_1, \beta_2 \in \mathfrak{t}^*_+$ and that $g_i x$ is λ_{β_i}-adapted for $i = 1, 2$. Then x is adapted for $g_i^{-1} \lambda_{\beta_i} g_i$ for $i = 1, 2$. By Theorem 2.1 (3) (b), $\lambda_{\beta_1}, \lambda_{\beta_2}$ are conjugate. Since $\lambda_{\beta_1}, \lambda_{\beta_2} \in \mathfrak{t}_+$, $\lambda_{\beta_1} = \lambda_{\beta_2}$. So $\beta_2 = a\beta_1$ for a positive rational number a. Since $\mu(x, \lambda_{\beta_i}) = \|\beta_i\|^2_{\mathfrak{t}^*}$ for $i = 1, 2$, $\beta_1 = \beta_2$. Therefore, the union in (3.7) is disjoint.

Suppose that $g_1, g_2 \in G$, $v_1, v_2 \in Y^{\mathrm{ss}}_\beta$ and that $g_1 v_1 = g_2 v_2$. Then $g_1 \pi_V(v_1) = g_2 \pi_V(v_2)$. Kirwan [8, 13.5.Theorem.] proved that $\pi_V(S_\beta) \cong G \times_{P_\beta} \mathbb{P}(Y_\beta)^{\mathrm{ss}}$. Therefore, there exists $p \in P_\beta$ such that $g_1 = g_2 p$, $\pi_V(v_2) = \pi_V(p v_1)$. So, there exists $c \in k^\times$ such that $v_2 = c p v_1$. Since $g_1 v_1 = g_2 v_2$, $g_2 p v_1 = g_2 c p v_1 = c g_2 p v_1$. Since $v_1 \neq 0$, $g_2 p v_1 \neq 0$. Therefore, $c = 1$. This implies that $S_\beta \cong G \times_{P_\beta} Y^{\mathrm{ss}}_\beta$. □

4. The non-split case

Assume that k is an arbitrary perfect field from now on. Therefore, $\bar{k} = k^{\mathrm{sep}}$. In this section $S \subset G$ is a maximal split torus and $S \subset T \subset G$ is a maximal torus defined over k.

We include the following two lemmas which are basically well-known for the sake of the reader. As we stated in Introduction, ${}_k\mathbb{W}$ can be regarded as a subgroup of $\mathbb{W}$ ([2, 5.5.Corollaire.]).

LEMMA 4.1. *Elements of $N_G(S)/Z_G(S)$ are represented by elements of $N_G(T)$.*

Proof. Let $n \in N_G(S)_{\bar{k}}$ (in fact, it is possible to choose a representative from $N_G(S)_k$). Then $n^{-1} T n, T$ are maximal tori containing S. So they are contained in $Z_G(S)$. Therefore, there exists an element $g \in Z_G(S)_{\bar{k}}$ such that $g^{-1} n^{-1} T n g = T$. This implies that $h \stackrel{\mathrm{def}}{=} ng \in N_G(T)_{\bar{k}}$. Then $n = hg^{-1} \in h Z_G(S)_{\bar{k}}$. So elements of of $N_G(S)/Z_G(S)$ are represented by elements of $N_G(T)$. □

LEMMA 4.2. *Suppose that λ_1, λ_2 are 1-PS's in S and that λ_1, λ_2 are conjugate in G_k, i.e., there exists $h \in G_k$ such that $h\lambda_1(t)h^{-1} = \lambda_2(t)$ ($t \in \bar{k}^\times$). Then λ_1, λ_2 are conjugate by an element of ${}_k\mathbb{W} = N_G(S)/Z_G(S)$.*

Proof. Let S_1, S_2 be the images of λ_1, λ_2 respectively. Then $h S_1 h^{-1} = S_2$. Since hSh^{-1}, S are both maximal split torus in $Z_G(S_2)$, there is $h_1 \in Z_G(S_2)$ such that $h_1 h S h^{-1} h_1^{-1} = S$. So $h_1 h \in N_G(S)$. Since $h\lambda_1(t)h^{-1} = \lambda_2(t)$ and $h_1 \in Z_G(S_2)$,

$h_1 h \lambda_1(t) h^{-1} h_1^{-1} = \lambda_2(t)$. Therefore, λ_1, λ_2 are conjugate by an element of the relative Weyl group ${}_k\mathbb{W}$. □

Proof of Theorem 1.3 *in the non-split case.* We now prove Theorem 1.3 in the non-split case.

We choose a coordinate system $v' = (v'_0, \ldots, v'_N)$ over $\bar{k}$ so that the action of T is diagonalized. Then the action of S is diagonalized also. However, since we chose a coordinate system $v = (v_0, \ldots, v_N)$ so that the action of S is diagonalized over k in Theorem 1.3, we consider the relation between the two coordinates v' and v.

The inclusion map $\mathfrak{s} \to \mathfrak{t}$ induces a map $\mathfrak{s}^* \to \mathfrak{t}^*$, which enables us to identify $\mathfrak{s}^*$ with a subspace of $\mathfrak{t}^*$. The restriction to $\mathfrak{s}^*$ induces a map $\mathfrak{t}^* \to \mathfrak{s}^*$. We show that this is the orthogonal projection.

If $\alpha \in \mathfrak{s}^*$ then $(\lambda(\alpha), \nu)_{\mathfrak{s}} = \langle \alpha, \nu \rangle_S$ for all $\nu \in \mathfrak{s}$. Regarding $\lambda(\alpha)$ as an element of $\mathfrak{t}$, the corresponding element of $\mathfrak{t}^*$ is the function g on $\mathfrak{t}$ such that $(\lambda(\alpha), \nu)_{\mathfrak{t}} = \langle g, \nu \rangle_T$ for all $\nu \in \mathfrak{t}$. If we restrict g to $\mathfrak{s}$, we obtain the function $\mathfrak{s} \ni \nu \mapsto (\lambda(\alpha), \nu)_{\mathfrak{t}} = \langle \alpha, \nu \rangle_S$. So the composition $\mathfrak{s}^* \to \mathfrak{t}^* \to \mathfrak{s}^*$ is the identity map. Therefore, if we denote the kernel of $\mathfrak{t}^* \to \mathfrak{s}^*$ by U then $\mathfrak{t}^* = \mathfrak{s}^* \oplus U$.

We show that U is orthogonal to $\mathfrak{s}^*$. If $\upsilon \in U$ then $\langle \upsilon, \lambda \rangle_T = 0$ for all $\lambda \in \mathfrak{s}$. If $\lambda \in \mathfrak{s}$ corresponds to $\beta(\lambda) \in \mathfrak{s}^*$ then $0 = \langle \upsilon, \lambda \rangle_T = (\upsilon, \beta(\lambda))_{\mathfrak{t}^*}$. Therefore, υ is orthogonal to $\mathfrak{s}^*$. This implies that $\mathfrak{t}^* \to \mathfrak{s}^*$ is the orthogonal projection.

Let $\gamma_i \in \mathfrak{s}^*$ (resp. $\eta'_i \in \mathfrak{t}^*$) be the weight of the i-th coordinate v_i (resp. v'_i). Note that since S is split, any character of S is defined over k. Let $\eta_i \in \mathfrak{s}^*$ be the restriction of η'_i to $\mathfrak{s}^*$. Removing the duplication, we put $\{\eta_0, \ldots, \eta_N\} = \{\delta_1, \ldots, \delta_m\}$ where $\delta_i \neq \delta_j$ if $i \neq j$. Let $A_i = \{j \mid \eta_j = \delta_i\}$. Then A_i is invariant under the action of $\mathrm{Gal}(\bar{k}/k)$. Let e_j (resp. e'_j) be the coordinate vector corresponding to the j-th coordinate v_j (resp. v'_j). Let $E'_i \subset V \otimes_k \bar{k}$ be the subspace spanned by $\{e'_j \mid j \in A_i\}$. Then E'_i is invariant under the action of $\mathrm{Gal}(\bar{k}/k)$. Since k is a perfect field, there exists a subspace $E_i \subset V$ such that $E'_i = E_i \otimes_k \bar{k}$. Since $V \otimes_k \bar{k} = \bigoplus_{i=1}^m E'_i$, we have $V = \bigoplus_{i=1}^m E_i$. This implies that E_i is the weight space of δ_i. Since the set of weights of V with respect to S does not depend on the choice of the coordinate, δ_i must coincides with γ_j for some j. So if we put $B_i = \{0 \leq j \leq N \mid \gamma_j = \delta_i\}$ then E_i is spanned by $\{e_j \mid j \in B_i\}$. Therefore, we can conclude that if η'_i is the weight of a non-zero coordinate of v' then η_i is the weight of a non-zero coordinate of v where v and v' are related with the change of coordinate.

Suppose that $x \in \mathbb{P}(V)_k \setminus \mathbb{P}(V)_k^{\mathrm{ss}}$ and that x is λ-adapted. Then λ is a 1-PS defined over k (by Theorem 2.3). So there exists $g \in G_k$ such that $g\lambda g^{-1}$ is a 1-PS in $\mathfrak{s}_+$. As in the split case, there is a positive rational number a such that if $\beta = a\beta(\lambda)$ then $gx \in Y_{\beta\,k}^{\mathrm{ss}}$. Here we have to make sure that the definition of $Y_{\beta\,k}^{\mathrm{ss}}$ is the same whether or not we regard $\beta \in \mathfrak{s}^*$ or $\beta \in \mathfrak{t}^*$.

To distinguish the difference between $k, \bar{k}$, let $Z'_\beta \subset V \otimes_k \bar{k}$ be the subspace spanned by e'_j such that $(\eta'_j, \beta)_{\mathfrak{t}^*} = \|\beta\|^2_{\mathfrak{t}^*}$. We define W'_β similarly. Let E_i, E'_i be as above. We regard $\beta \in \mathfrak{t}^*$. Since $\mathfrak{t}^* \to \mathfrak{s}^*$ is the orthogonal projection, $(\eta'_j, \beta)_{\mathfrak{t}^*} = (\eta_j, \beta)_{\mathfrak{s}^*}$. So Z'_β is spanned by E'_i such that $(\delta_i, \beta)_{\mathfrak{s}^*} = \|\beta\|^2_{\mathfrak{t}^*} = \|\beta\|^2_{\mathfrak{s}^*}$. Also Z_β is spanned by E_i such that $(\delta_i, \beta)_{\mathfrak{s}^*} = \|\beta\|^2_{\mathfrak{s}^*}$. Therefore, $Z'_\beta = Z_\beta \otimes_k \bar{k}$. Similarly we have $W'_\beta = W_\beta \otimes_k \bar{k}$.

We pointed out earlier that the notion of semistability does not depend on the ground field. Therefore, the set $Y^{\mathrm{ss}}_{\beta\,k}$ can be regarded as the set of k-rational points of the set Y^{ss}_{β} defined regarding $\beta \in \mathfrak{t}^*$.

By these considerations, we can conclude that $x \in Y^{\mathrm{ss}}_{\beta\,k}$ where the definition of $Y^{\mathrm{ss}}_{\beta\,k}$ is as in Introduction. We have to verify that this β belongs to $\mathfrak{B}$.

Put $x_1 = gx$. Suppose that $x_1 = \pi_V(v)$ where $v = (v_0, \ldots, v_N)$ (this is the coordinate for which the action of S is diagonalized rationally). Let $v' = (v'_0, \ldots, v'_N)$ be the coordinate of x_1 for which the action of T is diagonalized. We claim that this β is the closest point to the origin of the convex hull of $\mathfrak{I}_{x_1} = \{\gamma_j \mid v_j \neq 0\}$. We have already proved this claim in the split case. So β is the closest point to the origin of the convex hull of $\{\eta'_j \mid v'_j \neq 0\}$.

We have $\eta'_j = \eta_j + \varepsilon_j$ where ε_j is orthogonal to $\mathfrak{s}^*$. Let $\mathrm{pr} : \mathfrak{t}^* \to \mathfrak{s}^*$ be the natural map. Since $\beta \in \mathfrak{s}^*$, $\mathrm{pr}(\beta) = \beta$. Also since pr is a linear map and β is in the convex hull of $\{\eta'_j \mid v'_j \neq 0\}$, $\beta = \mathrm{pr}(\beta)$ is in the convex hull of $\{\mathrm{pr}(\eta'_j) \mid v'_j \neq 0\} = \{\eta_j \mid v_j \neq 0\}$. Since $\beta \in \mathfrak{s}^*$,

$$(\beta, \eta_i)_{\mathfrak{s}^*} = (\beta, \eta_i)_{\mathfrak{t}^*} = (\beta, \eta'_i)_{\mathfrak{t}^*} \geq \|\beta\|^2_{\mathfrak{t}^*} = \|\beta\|^2_{\mathfrak{s}^*}\,.$$

So β is the closest point to the origin of the convex hull of $\{\eta_j \mid v_i \neq 0\}$ in $\mathfrak{s}^*$.

We pointed out earlier that if η'_j is the weight of a non-zero coordinate (for which the action of T is diagonalized over $\bar{k}$) of x_1 then η_j coincides with the weight of a non-zero coordinate (for which the action of S is diagonalized over k) of x_1. Since $\beta \in \mathfrak{s}^*_+$, $\beta \in \mathfrak{B}$.

Finally, we have to prove that $S_{\beta\,k} \cong G_k \times_{P_{\beta\,k}} Y^{\mathrm{ss}}_{\beta\,k}$. Let $v \in S_{\beta\,k}$ and $x = \pi_V(v)$. Then any $\lambda \in \Lambda_x$ is split. So there exists $g \in G_k$ such that $gv \in Y^{\mathrm{ss}}_{\beta} \cap V_k = Y^{\mathrm{ss}}_{\beta\,k}$. Therefore, $G_k \times Y^{\mathrm{ss}}_{\beta\,k} \to S_{\beta\,k}$ is surjective. If $g_1, g_2 \in G_k$, $y_1, y_2 \in Y^{\mathrm{ss}}_{\beta\,k}$ and $v = g_1 y_1 = g_2 y_2$ then there exists $h \in P_{\beta\,\bar{k}}$ such that $g_1 = g_2 h$ by the split case. But then $h = g_1 g_2^{-1} \in G_k$ and so $h \in P_{\beta\,k}$. Therefore, $S_{\beta\,k} \cong G_k \times_{P_{\beta\,k}} Y^{\mathrm{ss}}_{\beta\,k}$. □

Proof of Corollary 1.4. Let G, G_1, etc., be as in the situation of Corollary 1.4. We remind the reader that we are considering the stability with respect to the group G_1 (not G).

Let $\beta \in \mathfrak{B}$. We put $P_{1\beta} = P_\beta \cap G_1$. We have proved that $S_{\beta\,k} \cong G_{1\,k} \times_{P_{1\beta\,k}} Y^{\mathrm{ss}}_{\beta\,k}$. So the map $G_k \times_{P_{\beta\,k}} Y^{\mathrm{ss}}_{\beta\,k} \to S_{\beta\,k}$ is surjective. Suppose that $g_1, g_2 \in G_k, v_1, v_2 \in Y^{\mathrm{ss}}_{\beta\,k}$ and that $g_1 v_1 = g_2 v_2$. This implies that $g_2^{-1} g_1 v_1 = v_2$. Since $G_{\bar{k}} = T_{0\,\bar{k}} G_{1\,\bar{k}}$, there exist $t \in T_{0\,\bar{k}}, h \in G_{1\,\bar{k}}$ such that $g_2^{-1} g_1 = th$. Since $t \in T_{0\,\bar{k}} \subset Z(G)_{\bar{k}}$, $thv_1 = htv_1 = h\chi(t)v_1 = v_2$. Since Y^{ss}_{β} is invariant under scalar multiplications, we have $\chi(t)v_1 \in Y^{\mathrm{ss}}_{\beta\,\bar{k}}$. This implies that $h \in P_{1\beta\,\bar{k}} \subset P_{\beta\,\bar{k}}$. Since $T_0 \subset Z(G)$, $T_0 \subset P_\beta$. So $g_2^{-1} g_1 = th \in P_{\beta\,\bar{k}}$. Since $g_1, g_2 \in G_k$, we have $h \in P_{\beta\,k}$. Therefore, $S_{\beta\,k} \cong G_k \times_{P_{\beta\,k}} Y^{\mathrm{ss}}_{\beta\,k}$.

Other statements follow trivially from Theorem 1.3. □

5. Examples

Let k be a perfect field, k_1/k a quadratic extension, D a division quaternion algebra over k, $\mathrm{H}_3(k_1)$ the space of 3×3 Hermitian matrices with entries in k_1 (with respect to the action of $\mathrm{Gal}(k_1/k)$) and $\mathrm{H}_3(D)$ the space of 3×3 Hermitian matrices with entries in D

(with respect to the canonical involution of D). Let $\mathbb{O}$ be a non-split octonion. For example, if k is a number field with a real place v and D is a division quaternion algebra such that $D \otimes k_v$ is isomorphic to the Hamiltonian quaternions $\mathbb{H}$, then the algebra $D(+)$ obtained from the Cayley–Dickson process (see [3, pp.101–110]) is a non-split octonion. Let J be the exceptional Jordan algebra of 3×3 Hermitian matrices with entries in $\mathbb{O}$. There is a well-defined "determinant" Δ on J. Let $E_6 = \{g \in \mathrm{GL}(J) \mid \Delta(gx) = \Delta(x) \text{ for all } x \in J\}$. This is a simple group of type E_6 with split rank 2.

We first consider the following four prehomogeneous vector spaces.

(a) $G = \mathrm{GL}_3 \times \mathrm{GL}_2$, $V = \mathrm{Sym}^2\mathrm{Aff}^3 \otimes \mathrm{Aff}^2$.
(b) $G = \mathrm{Res}_{k_1/k}\,\mathrm{GL}_3 \times \mathrm{GL}_2$, $V = \mathrm{H}_3(k_1) \otimes \mathrm{Aff}^2$.
(c) $G = \mathrm{GL}_3(D) \times \mathrm{GL}_2$, $V = \mathrm{H}_3(D) \otimes \mathrm{Aff}^2$.
(d) $G = E_6 \times \mathrm{GL}_2$, $V = J \otimes \mathrm{Aff}^2$.

These four representations are prehomogeneous vector spaces of parabolic type coming from simple groups of types F_4, E_6, E_7, E_8 respectively.

Rational orbits for the cases (a)–(c) have interesting arithmetic interpretations. Such interpretations were discussed in [16], [4], [15] for the cases (a)–(c) respectively. However, the interpretation for the case (d) is unknown.

They have exactly the same set of weights as follows.

Let G_1 be $\mathrm{SL}_3 \times \mathrm{SL}_2$, $\mathrm{Res}_{k_1/k}\,\mathrm{SL}_3 \times \mathrm{SL}_2$ and $E_6 \times \mathrm{SL}_2$ respectively for the cases (a), (b), (d). For the case (c), $\mathrm{GL}_3(D)$ can be identified with a subgroup of GL_{12}. Let $d : \mathrm{GL}_{12} \to \mathrm{GL}_1$ be the determinant and $G_1 = \mathrm{Ker}(d)^\circ \times \mathrm{GL}_2$. We use G_1 for the G_1 in Corollary 1.4. In all four cases, let S_1 be the set of diagonal matrices of SL_3, $\mathrm{Res}_{k_1/k}\,\mathrm{SL}_3$, $\mathrm{SL}_3(D)$, E_6 with entries in $k^\times$ respectively and S_2 the set of diagonal matrices in SL_2. Then $S = S_1 \times S_2$ is a maximal split torus of G_1. Let $T_0 = \mathrm{GL}_1 \times \mathrm{GL}_1$, $\mathrm{Res}_{k_1/k}\,\mathrm{GL}_1 \times \mathrm{GL}_1$, $\mathrm{GL}_1 \times \mathrm{GL}_1$ and GL_1 respectively for the cases (a)–(d). In the cases (a), (d), two factors of GL_1 are subgroups of diagonal matrices with entries in $k^\times$. The case (b) is similar. In the case (d), E_6 is simple and GL_1 is the subgroup of diagonal matrices in GL_2. Then we are in the situation of Corollary 1.4.

Let $a_n(t_1, \ldots, t_n)$ be the diagonal matrix with diagonal entries $t_1, \ldots, t_n \in k^\times$. For $t_{11}, t_{12}, t_{13} \in k^\times$ and $t_{21}, t_{22} \in k^\times$, we put

$$t_1 = a_3(t_{11}, t_{12}, t_{13}), \quad t_2 = a_3(t_{21}, t_{22}), \quad t = (t_1, t_2) \in S.$$

We identify $\mathfrak{s}^*$ with

$$\{z = (z_{11}, z_{12}, z_{13}; z_{21}, z_{22}) \in \mathbb{R}^5 \mid z_{11} + z_{12} + z_{13} = 0,\ z_{21} + z_{22} = 0\}.$$

We use the notation $z_1 = (z_{11}, z_{12}, z_{13})$, $z_2 = (z_{21}, z_{22})$, $z = (z_1, z_2)$. For

$$z = (z_1, z_2), \quad z' = (z_1', z_2') \quad (z_1 = (z_{11}, z_{12}, z_{13}), \quad z_2 = (z_{21}, z_{22}) \text{ and similarly for } z')$$

we define

$$(z, z')_{\mathfrak{s}^*} = z_{11}z_{11}' + z_{12}z_{12}' + z_{13}z_{13}' + z_{21}z_{21}' + z_{22}z_{22}'.$$

This inner product is Weyl group invariant. Let $\|\ \|_{\mathfrak{s}^*}$ be the metric defined by this bilinear form. We choose

$$\mathfrak{s}_+^* = \{(z_{11}, z_{12}, z_{13}; z_{21}, z_{22}) \mid z_{11} \le z_{12} \le z_{13}, z_{21} \le z_{22}\}$$

as the Weyl chamber.

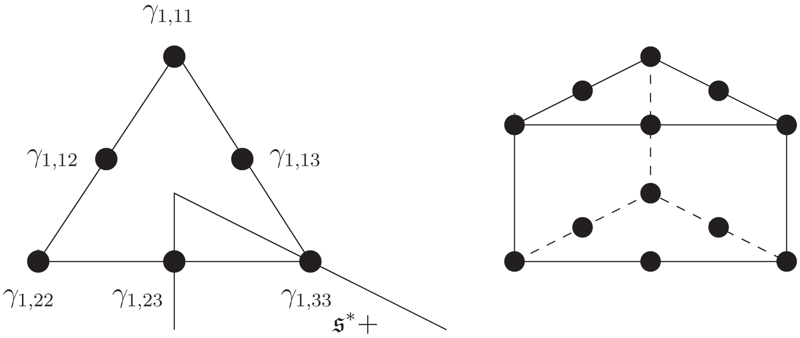


FIGURE 1.

We define $\gamma_{i,jk}$ as follows.

$\gamma_{1,11}$	$\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; \frac{1}{2}, -\frac{1}{2}\right)$	$\gamma_{2,11}$	$\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; -\frac{1}{2}, \frac{1}{2}\right)$
$\gamma_{1,12}$	$\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; \frac{1}{2}, -\frac{1}{2}\right)$	$\gamma_{2,12}$	$\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; -\frac{1}{2}, \frac{1}{2}\right)$
$\gamma_{1,13}$	$\left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}; \frac{1}{2}, -\frac{1}{2}\right)$	$\gamma_{2,13}$	$\left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}; -\frac{1}{2}, \frac{1}{2}\right)$
$\gamma_{1,22}$	$\left(-\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}; \frac{1}{2}, -\frac{1}{2}\right)$	$\gamma_{2,22}$	$\left(-\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}; -\frac{1}{2}, \frac{1}{2}\right)$
$\gamma_{1,23}$	$\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{2}, -\frac{1}{2}\right)$	$\gamma_{2,23}$	$\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}; -\frac{1}{2}, \frac{1}{2}\right)$
$\gamma_{1,33}$	$\left(-\frac{2}{3}, -\frac{2}{3}, \frac{4}{3}; \frac{1}{2}, -\frac{1}{2}\right)$	$\gamma_{2,33}$	$\left(-\frac{2}{3}, -\frac{2}{3}, \frac{4}{3}; -\frac{1}{2}, \frac{1}{2}\right)$

Then with our metric, we can express $\gamma_{i,jk}$'s as above. These are the weights of coordinates of V. The picture on the right shows the weights of V and the weights of the upper half are shown in the picture on the left. In the case (a), $\gamma_{i,jk}$ corresponds to the monomial $v_j v_k$ of three variables v_1, v_2, v_3. The (z_{11}, z_{12}, z_{13}) part of the Weyl chamber $\mathfrak{s}^*_+$ is the lower right region as above. The (z_{21}, z_{22}) part of $\mathfrak{s}^*_+$ is the lower half of the vertical line. So $\mathfrak{s}^*_+$ consists of vectors which point down and right and coming toward the reader in the picture on the right.

The set $\mathfrak{B}$ corresponds to the 12 convex hulls in the next page. The case (a) is discussed in [17, pp. 198–205] and so we do not include the details here. $S_{\beta_{11}}$, $S_{\beta_{12}}$ are the empty set in all cases and so the cases (a)–(d) all have 10 unstable strata.

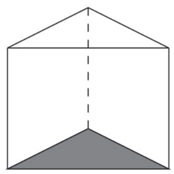
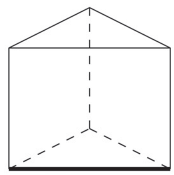
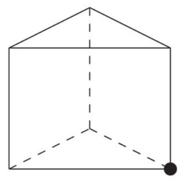
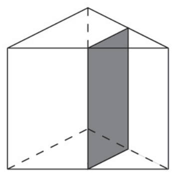
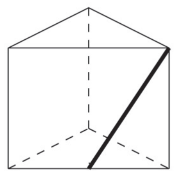
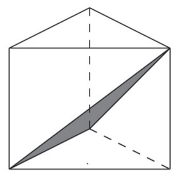
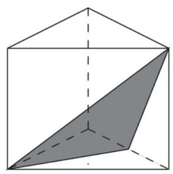
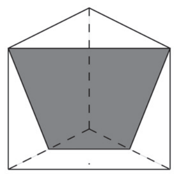
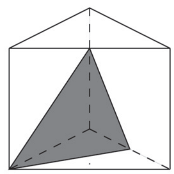
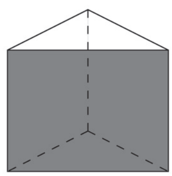
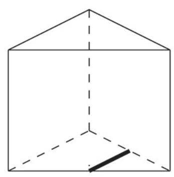
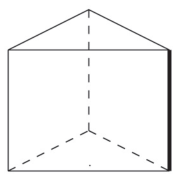
There are smaller and easier prehomogeneous vector spaces similar to the above examples which have the same set of weights. Consider the following prehomogeneous vector spaces.

(a′) $G = \mathrm{GL}_2 \times \mathrm{GL}_2$, $V = \mathrm{Sym}^2 \mathrm{Aff}^2 \otimes \mathrm{Aff}^2$.

(b′) $G = \mathrm{Res}_{k_1/k}\, \mathrm{GL}_2 \times \mathrm{GL}_2$, $V = \mathrm{H}_2(k_1) \otimes \mathrm{Aff}^2$.

(c′) $G = \mathrm{GL}_2(D) \times \mathrm{GL}_2$, $V = \mathrm{H}_2(D) \otimes \mathrm{Aff}^2$.

There does not seem to be an analogue of the case (d) above, but there are more cases with the same weights as the cases (a′)–(c′). The case (a′) is essentially the same as the space of (single) binary quadratic forms by the castling transformation (see [13, p. 39]). The global zeta function for the case (a′) was needed to determine the pole structure of the global zeta function for the case (a) (see [17]). The global zeta function for the case

strata	convex hull	strata	convex hull	strata	convex hull
S_{β_1}		S_{β_2}		S_{β_3}	
S_{β_4}		S_{β_5}		S_{β_6}	
S_{β_7}		S_{β_8}		S_{β_9}	
$S_{\beta_{10}}$		$S_{\beta_{11}}$		$S_{\beta_{12}}$	

(b′) was considered in [18]. Note that in [18], the structure of $V_k \setminus V_k^{\mathrm{ss}}$ was considered explicitly and by Theorem 1.3, it can now be replaced by the convex hull consideration in [17, pp. 153–154]. The density theorem related to the case (b′) was proved in [5], [6]. The interpretation of rational orbits of the case (c′) was considered in [15].

Let G_1 be $\mathrm{SL}_2 \times \mathrm{SL}_2$, $\mathrm{Res}_{k_1/k}\,\mathrm{SL}_2 \times \mathrm{SL}_2$ for the cases (a′), (b′) respectively. $\mathrm{GL}_2(D)$ can be identified with a subgroup of GL_8. Let $d : \mathrm{GL}_8 \to \mathrm{GL}_1$ be the determinant and $G_1 = \mathrm{Ker}(d)^\circ \times \mathrm{GL}_2$. We use G_1 for the G_1 in Corollary 1.4. In all three cases, let S_1 be the set of diagonal matrices with entries in $k^\times$ of GL_2, $\mathrm{Res}_{k_1/k}\,\mathrm{GL}_2$, $\mathrm{GL}_2(D)$ respectively and S_2 the set of diagonal matrices of SL_2. Then $S = S_1 \times S_2$ is a maximal split torus of G_1. Let $T_0 = \mathrm{GL}_1 \times \mathrm{GL}_1$, $\mathrm{Res}_{k_1/k}\,\mathrm{GL}_1 \times \mathrm{GL}_1$ $\mathrm{GL}_1 \times \mathrm{GL}_1$ for the cases (a′)–(c′) respectively where two factors are the subgroups of diagonal matrices. Then we are in the situation of Corollary 1.4.

We identify $\mathfrak{s}^*$ with

$$\{z = (z_{11}, z_{12}; z_{21}, z_{22}) \in \mathbb{R}^4 \mid z_{11} + z_{12} = 0,\ z_{21} + z_{22} = 0\}\,.$$

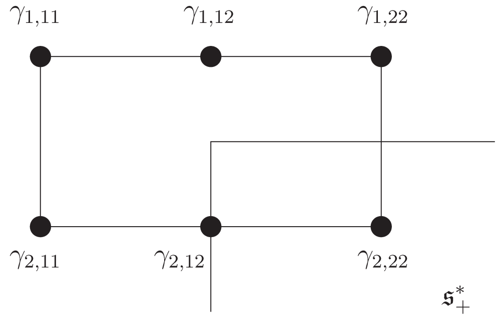
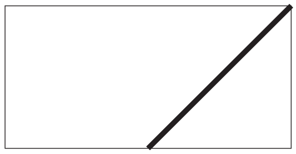
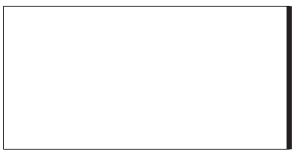
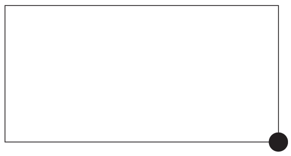


FIGURE 2.

strata	convex hull	strata	convex hull
S_{β_1}		S_{β_2}	
S_{β_3}		S_{β_4}	

We choose

$$\mathfrak{s}_+^* = \{(z_{11}, z_{12}; z_{21}, z_{22}) \mid z_{11} \leq z_{12}, z_{21} \leq z_{22}\}$$

as the Weyl chamber. We use a similar inner product on $\mathfrak{s}$, $\mathfrak{s}^*$ as the cases (a)–(d).

We define $\gamma_{i,jk}$ as follows.

$$\begin{array}{llll} \gamma_{1,11} & \left(1, -1; \frac{1}{2}, -\frac{1}{2}\right) & \gamma_{2,11} & \left(1, -1; -\frac{1}{2}, \frac{1}{2}\right) \\ \gamma_{1,12} & \left(0, 0; \frac{1}{2}, -\frac{1}{2}\right) & \gamma_{2,12} & \left(0, 0; -\frac{1}{2}, \frac{1}{2}\right) \\ \gamma_{1,22} & \left(-1, 1; \frac{1}{2}, -\frac{1}{2}\right) & \gamma_{2,22} & \left(-1, 1; -\frac{1}{2}, \frac{1}{2}\right) \end{array}$$

Then with our metric, we can express $\gamma_{i,jk}$'s as in Figure 2.

These are the weights of coordinates of V. The Weyl chamber $\mathfrak{s}_+^*$ is the lower right region as above. The set $\mathfrak{B}$ corresponds to the 4 convex hulls in the above table. The stratum S_{β_3} is the empty set in all cases and so the cases (a′), (b′), (c′) all have 3 unstable strata.

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